Math 2040 C Week 9

Orthonormal Bases

Let V be an inner product space, $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$

Defn 6.23, 6.27 $v_1,\cdots v_m \in V$ are called

① orthogonal if $v_i \perp v_j$ for any $i \neq j$

② orthonormal if they are orthogonal and each v_i is a unit vector (i.e. $\|v_i\|=1$)

③ an orthonormal basis of V if they are orthonormal and form a basis

Rmk ① We sometimes write $e_1,\cdots,e_m$ to denote orthonormal vectors. Note that

$$\langle e_i, e_j\rangle = \underbrace{\delta_{ij}}_{\text{Called Kronecker delta}} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

② These concepts can be defined similarly for subsets,

e.g. A subset $S\subseteq V$ is called orthogonal if $v\perp w$ for any distinct $v,w\in S$.

eg

① $\{e_1,\cdots,e_n\}$ is an orthonormal basis for $\mathbb{F}^n$

② $\left\{\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{2}}\\ -\frac{i}{\sqrt{2}}\end{bmatrix}\right\}$ is an orthonormal basis for $\mathbb{C}^2$

③ $\{1, x\}$ is orthogonal under $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

Prop 5.25 If $e_1,\cdots,e_m\in V$ are orthonormal and $v=c_1e_1+\cdots+c_me_m$. Then

$$\|v\|^2 = |c_1|^2+\cdots+|c_m|^2$$

Pf Repeated application of Pythagorean theorem

Prop 6.26
An orthonormal list/set is linearly indept

Pf Let $e_1, \cdots, e_m$ be orthonormal. Suppose

$$c_1 e_1 + \cdots + c_m e_m = \vec{0}$$

Then for any i,

$$0 = \langle \vec{0}, e_i \rangle = \left\langle \sum_{j=1}^{m} c_j e_j, e_i \right\rangle = \sum_{j=1}^{m} c_j \langle e_j, e_i \rangle = c_i$$

$\therefore$ $e_1, \cdots, e_m$ are lin. indept.

Cor 5.28 Let $\dim V = n$. Then an orthonormal set of n vectors is an orthonormal basis.

Pf The orthonormal set consists of n linearly indept vectors in V, where $\dim V = n$, and so is a basis.

It is easy to calculate coefficients in a linear combination of orthonormal basis:

Prop 6.30 Let $e_1, \cdots, e_n$ be an orthonormal basis of V and $v \in V$. Then

① $v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$

② $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$

Pf Since $e_1, \cdots, e_n$ is a basis, $\exists\, c_1, \cdots, c_n$ s.t.

$$v = c_1 e_1 + \cdots + c_n e_n$$

$$\Rightarrow \langle v, e_i \rangle = \left\langle \sum_{j=1}^{m} c_j e_j, e_i \right\rangle = \sum_{j=1}^{m} c_j \langle e_j, e_i \rangle = c_i$$

$\Rightarrow$ ①

② follows from ① and Prop 5.25

Orthonormal basis is useful.

How to find them?

Prop 6.31 (Gram-Schmidt Process)

Suppose $v_1, \cdots, v_m$ are lin. indept.

Define u_i and e_i as follows.

① $u_1 = v_1$ and $e_1 = \frac{u_1}{\|u_1\|}$

② For $j = 2, \cdots, m$, inductively define

$$u_j = v_j - \sum_{i=1}^{j-1} \langle v_j, e_i \rangle e_i \quad \text{and} \quad e_j = \frac{u_j}{\|u_j\|}$$

Then $e_1, \cdots, e_m$ are orthonormal.

Also, for $1 \le j \le m$,

$$\operatorname{span}\{v_1, \cdots, v_j\} = \operatorname{span}\{e_1, \cdots, e_j\}$$

Rmk ① In the formula

$$u_j = v_j - \underbrace{\sum_{i=1}^{j-1} \langle v_j, e_i \rangle e_i}_{\text{Orthogonal Projection of } v_j \text{ onto span}\{e_1, \cdots, e_{j-1}\}}$$

Orthogonal Projection of v_j onto span$\{e_1, \cdots, e_{j-1}\}$ discussed in section 6C.

② The process $e_i = \frac{u_i}{\|u_i\|}$ is called normalization

Gram Schmidt Process

$\boxed{\text{lin indept}}$ $\quad v_1 \quad v_2 \quad v_3 \quad \cdots \quad v_{n-1} \quad v_n$

↓ orthagonalization

$\boxed{\text{orthogonal}}$ $\quad u_1 \quad u_2 \quad u_3 \quad \cdots \quad u_{n-1} \quad u_n$

↓ normalization $\quad$ steps

$\boxed{\text{orthonormal}}$ $\quad e_1 \quad e_2 \quad e_3 \quad \cdots \quad e_{n-1} \quad e_n$

Pf We prove inductively on $j=1,\cdots,m$ that

$$(*)\ \begin{cases} e_1,\cdots,e_j \text{ are orthonormal} \\ \operatorname{span}\{e_1,\cdots,e_j\} = \operatorname{span}\{v_1,\cdots,v_j\} \end{cases}$$

Clearly it is true for $j=1$: $e_1 = \dfrac{v_1}{\|v_1\|}$

Assume we proved $(*)$ for $j=k$

Then for $j=k+1$

note $v_{k+1} \notin \operatorname{span}\{v_1,\cdots,v_k\}$

$$= \operatorname{span}\{e_1,\cdots,e_k\}$$

$$\Rightarrow u_{k+1} = v_{k+1} - \sum_{i=1}^{k} \langle v_{k+1}, e_i\rangle e_i \neq \vec{0}$$

$\therefore\ e_{k+1} = \dfrac{u_{k+1}}{\|u_{k+1}\|}$ is well-defined,

$$\|e_{k+1}\| = 1$$

Also, for $1 \le \ell \le k$

$$\begin{aligned} \langle e_{k+1}, e_\ell\rangle &= \frac{1}{\|u_{k+1}\|}\langle u_{k+1}, e_\ell\rangle \\ &= \frac{1}{\|u_{k+1}\|}\Big\langle v_{k+1} - \sum_{i=1}^{k}\langle v_{k+1}, e_i\rangle e_i, e_\ell\Big\rangle \\ &= \frac{1}{\|u_{k+1}\|}\Big(\langle v_{k+1}, e_\ell\rangle - \langle v_{k+1}, e_\ell\rangle\langle e_\ell, e_\ell\rangle\Big) \\ &= 0 \end{aligned}$$

Clearly $u_{k+1} \in \operatorname{span}\{e_1,\cdots,e_k, v_{k+1}\}$

$$= \operatorname{span}\{v_1,\cdots,v_k, v_{k+1}\}$$

Hence $\operatorname{Span}\{e_1,\cdots,e_k,e_{k+1}\} = \operatorname{span}\{v_1,\cdots,v_k,u_{k+1}\}$

$$\subseteq \operatorname{span}\{v_1,\cdots,v_k,v_{k+1}\}$$

$e_1,\cdots,e_{k+1}$ are orthonormal $\Rightarrow$ lin indept

$\therefore$ Both span have dim. $k+1$

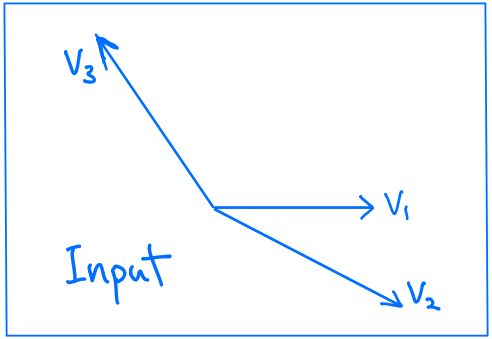
$\therefore$ The two span are the same

$(*)$ is true for $j=k+1$.

The proposition is proved by induction.

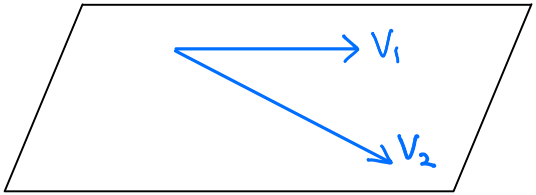
Picture $n=3$, $\mathbb{F}=\mathbb{R}$

Given lin indept v_1, v_2, v_3



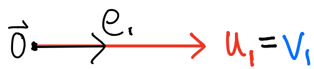
span$\{v_1\}$

span$\{v_1, v_2\}$

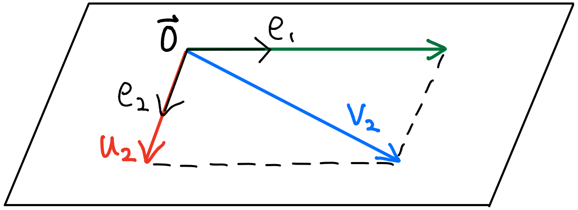


Gram-Schmidt Process

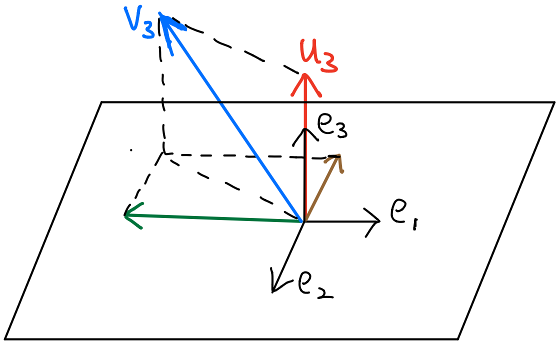
① $u_1 = v_1 \qquad e_1 = \dfrac{u_1}{\|u_1\|}$

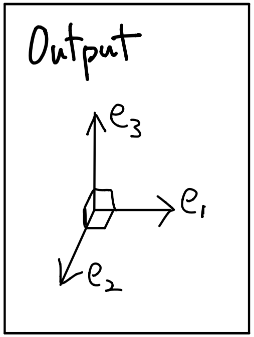


② $u_2 = v_2 - \langle v_2, e_1\rangle e_1, \qquad e_2 = \dfrac{u_2}{\|u_2\|}$



③ $u_3 = v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2, \qquad e_3 = \dfrac{u_3}{\|u_3\|}$





eg Find an orthonormal basis for $P_2(\mathbb{R})$ with inner product $\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

Pf $\alpha = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$.

Apply Gram-Schmidt Process to α:

let $v_1 = 1$, $v_2 = x$, $v_3 = x^2$.

Then $u_1 = v_1 = 1$, $\|u_1\|^2 = \int_{-1}^{1} 1^2\,dx = 2$

$$e_1 = \frac{u_1}{\|u_1\|} = \sqrt{\frac{1}{2}}$$

$$u_2 = v_2 - \langle v_2, e_1\rangle$$

$$= v_2 - \left(\int_{-1}^{1}(x)\left(\tfrac{1}{\sqrt{2}}\right)dx\right)e_1$$

$$= v_2 - 0\,e_1$$

$$= x$$

$$\|u_2\|^2 = \int_{-1}^{1} x^2\,dx = \frac{2}{3}$$

$$e_2 = \frac{u_2}{\|u_2\|} = \sqrt{\frac{3}{2}}\,x$$

$$u_3 = v_3 - \langle v_3, e_1\rangle e_1 - \langle v_3, e_2\rangle e_2$$

$$= x^2 - \left(\int_{-1}^{1} x^2\left(\sqrt{\tfrac{1}{2}}\right)dx\right)e_1 - \left[\int_{-1}^{1} x^2\left(\sqrt{\tfrac{3}{2}}\,x\right)dx\right]e_2$$

$$= x^2 - \left(\frac{2}{3}\cdot\sqrt{\frac{1}{2}}\right)\cdot\sqrt{\frac{1}{2}} - 0\cdot e_2$$

$$= x^2 - \frac{1}{3}$$

$$\|u_3\|^2 = \int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx = \int_{-1}^{1}\left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right)dx$$

$$= \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{8}{45}$$

$$e_3 = \frac{u_3}{\|u_3\|} = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$

$\therefore$ An orthonormal basis of $P_2(\mathbb{R})$ is

$$\left\{\sqrt{\frac{1}{2}},\ \sqrt{\frac{3}{2}}\,x,\ \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)\right\}$$

Consequences of Gram-Schmidt Process :

Prop 6.34 Every finite dim. inner product space has an orthonormal basis.

Pf Let V be a finite dim inner product space.

Choose a basis α of V, apply Gram-Schmidt Process to α to get an orthonormal set β.

Then β is lin. indept and $\text{span}\,\beta = \text{span}\,\alpha = V$

Hence β is an orthonormal basis of V

Prop 6.35 Let $\dim V < \infty$. $S \subseteq V$ be orthonormal. Then $\exists$ orthonormal basis S' of V with $S \subseteq S'$.

Pf Let $S = \{e_1, \cdots, e_m\}$. Extend it to a basis

$S'' = \{e_1, \cdots, e_m, v_{m+1}, \cdots v_n\}$ of V.

Apply Gram-Schmidt Process to S'' to get

an orthonormal basis $S' = \{f_1, \cdots, f_n\}$ of V

Note that $e_1, \cdots, e_m$ are already orthonormal.

From the formula in Gram-Schmidt Process,

$f_i = e_i$ for $i = 1, \cdots, m$.

$\therefore\ S \subseteq S' = \{e_1, \cdots, e_m, f_{m+1}, \cdots, f_n\}$

Lemma 6.37 Suppose $T \in L(V)$ and V has an ordered basis α, $M(T,\alpha)$ is upper triangular. Then $\exists$ orthonormal basis β of V s.t. $M(T,\beta)$ is upper triangular.

Pf Let $\alpha = \{v_1, \cdots, v_n\}$.

Apply Gram-Schmidt Process to α to get an orthonormal basis $\beta = \{e_1, \cdots, e_n\}$

By 5.26, $M(T,\alpha)$ is upper triangular.

$\Rightarrow \text{span}\{v_1, \cdots, v_j\}$ is T-invariant $\forall j$

$\therefore \text{span}\{e_1, \cdots, e_j\} = \text{span}\{v_1, \cdots, v_j\}$ is T-invariant.

By 5.26, $M(T,\beta)$ is upper triangular too

Thm 6.38 (Schur's Theorem)

Let $\dim V < \infty$, $\mathbb{F} = \mathbb{C}$, $T \in L(V)$

Then $M(T,\beta)$ is upper triangular for some orthonormal basis of V

Pf By 5.27, $\exists$ ordered basis α of V s.t. $M(T,\alpha)$ is upper triangular.

By 6.37, $\exists$ orthonormal basis β s.t. $M(T,\beta)$ is upper triangular

Linear Functional

Defn 6.39 Let V be a vector space$/\mathbb{F}$.

A <u>linear functional</u> on V is an element in $\mathcal{L}(V,\mathbb{F})$.

i.e. a linear map from V to $\mathbb{F}$

eg $\varphi : P_3(\mathbb{R}) \to \mathbb{R}$, $\varphi(p) = \int_0^1 p(t)e^t \, dt$

Defn Let V be an inner product space, $u \in V$

Define linear functional $\varphi_u \in \mathcal{L}(V, \mathbb{F})$ by

$$\varphi_u(v) = \langle v, u \rangle$$

eg $u = (2, i, 1-i) \in \mathbb{C}^3$. Then

$\varphi_u : \mathbb{C}^3 \longrightarrow \mathbb{C}$ with

$$\varphi_u(z_1, z_2, z_3) = \langle (z_1, z_2, z_3), (2, i, 1-i) \rangle$$
$$= 2z_1 - i z_2 + (1-i) z_3$$

Prop Let V be an inner product space

The map $\Phi : V \to \mathcal{L}(V, \mathbb{F})$ defined by

$\Phi(u) = \varphi_u$ is conjugate linear, i.e.

① $\Phi(u+w) = \Phi(u) + \Phi(v)$

② $\Phi(\alpha u) = \bar{\alpha} \Phi(u)$

Pf It follows from the fact that inner product is conjugate linear in 2nd slot

Rmk

If $\mathbb{F} = \mathbb{R}$, conjugate linear $=$ linear

If $\mathbb{F} = \mathbb{C}$, then Φ is linear over $\mathbb{R}$:

$$\Phi : V \longrightarrow \mathcal{L}(V, \mathbb{C})$$

Regarded as real vector space

$\boxed{\begin{array}{l}\underline{\text{Thm 6.42}} \text{ (Riesz Representation theorem)} \\ \text{Let } V \text{ be an inner product space, } \dim V < \infty \\ \varphi \in L(V, \mathbb{F}). \text{ Then } \exists \text{ unique } u \in V \text{ s.t.} \\ \qquad \varphi(v) = \langle v, u \rangle \quad \forall v \in V \end{array}}$

<u>Pf</u> Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of V

① <u>Existence of u</u>

Let $\boxed{u = \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n}$

For any $v \in V$,

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

$$\begin{aligned} \therefore \varphi(v) &= \langle v, e_1 \rangle \varphi(e_1) + \cdots + \langle v, e_n \rangle \varphi(e_n) \\ &= \langle v, \overline{\varphi(e_1)} e_1 \rangle + \cdots + \langle v, \overline{\varphi(e_n)} e_n \rangle \\ &= \langle v, \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n \rangle \\ &= \langle v, u \rangle \end{aligned}$$

② <u>Uniqueness of u</u> Suppose $u, w \in V$ s.t.

$\varphi(v) = \langle v, u \rangle = \langle v, w \rangle \quad \forall v \in V$

Then $\langle v, u - w \rangle = 0 \quad \forall v \in V$

Put $v = u - w$, then $\langle u - w, u - w \rangle = 0$

$\Rightarrow u - w = 0 \Rightarrow u = w$

<u>Alt Pf</u> (Non-constructive)

u exists and is unique $\iff$ $\Phi : V \to L(V, \mathbb{F})$, $\Phi(u) = \varphi_u$ is a bijection

Note ① Φ is linear over $\mathbb{R}$

② $\dim_{\mathbb{R}} V = \dim_{\mathbb{R}} L(V, \mathbb{F})$

③ Φ is injective : If $\Phi(u) = \varphi_u = T_0$

Then $\langle u, u \rangle = \varphi_u(u) = T_0(u) = 0$

$\Rightarrow u = 0$

$\therefore \Phi$ is a isomorphism over $\mathbb{R}$

$\Rightarrow \Phi$ is a bijection

eg Let $V = P_2(\mathbb{R}) \subseteq C([-1,1])$, $\varphi \in L(V, \mathbb{R})$.

$$\varphi(p) = \int_{-1}^{1} p(t)\cos(\pi t)\,dt$$

By Riesz Representation theorem,

$\exists$ unique $q \in P_2(\mathbb{R})$ s.t.

$$\varphi(p) = \langle p, q\rangle = \int_{-1}^{1} p(t) q(t)\,dt \quad \forall p \in P_2(\mathbb{R})$$

To find such q, consider the orthonormal basis

$$\underbrace{\sqrt{\tfrac{1}{2}}}_{e_1},\ \underbrace{\sqrt{\tfrac{3}{2}}\,x}_{e_2},\ \underbrace{\sqrt{\tfrac{45}{8}}\left(x^2 - \tfrac{1}{3}\right)}_{e_3}$$

Then from pf of Reisz Representation thm,

$$q(x) = \overline{\varphi(e_1)}\,e_1 + \overline{\varphi(e_2)}\,e_2 + \overline{\varphi(e_3)}\,e_3$$

$$= -\frac{45}{2\pi^2}\left(x^2 - \frac{1}{3}\right)$$

$\therefore$ For any $p \in P_2(\mathbb{R})$

$$\int_{-1}^{1} p(t)\cos(\pi t)\,dt = \langle p, q\rangle$$

$$= -\frac{45}{2\pi^2}\int_{-1}^{1} p(t)\left(t^2 - \frac{1}{3}\right)dt$$